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AN ANALYTIC APPROACH TO THE COLLATZ $3N + 1$ PROBLEM

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Abstract. Berg and Meinardus, 1994, 1995, introduced a pair of linear functional equations, acting on a space of holomorphic functions $\mathcal{H}$, defined on the open unit disk $\mathbb{D}$. The simultaneous solutions of the two functional equations contain a simple two dimensional space, denoted by Δ_2, and if one could show that Δ_2 is the only solution, the Collatz conjecture would be true. Berg and Meinardus already presented the general solutions of the two individual functional equations. The present author reformulates the pair of functional equations in form of two linear operators, denoted by U and V. Thus, $\mathcal{K} := \{h \in \mathcal{H} : U[h] = 0, V[h] = 0\}$ is of main interest. Since the general solutions of $U[h] = 0$, $V[h] = 0$, denoted by $\mathcal{K}_U, \mathcal{K}_V$, respectively, are already known, we compute $U[h]$ for $h \in \mathcal{K}_V$ and study the consequences of $U[h] = 0$. We show, that, indeed $\Delta_2 = \mathcal{K}$ follows, which implies that the Collatz conjecture is true.

Key words. Collatz problem, $3n+1$ problem, linear operators acting on holomorphic functions.

AMS subject classifications. 11B37, 11B83, 30D05, 39B32, 39B62.

1. Introduction: The Collatz or $3n + 1$ problem. Since the problem is very well investigated, one source is Wirsching, 2000, [7], another one is Collatz, 1986, [3], and a new comprehensive, source is a book edited by Lagarias, 2010, [5], we only state the problem in short. Throughout the paper, let $\mathbb{N}$ be the set of positive integers. We define the following sequences $\mathbb{N} \cup \{0\} \to \mathbb{N}$, to be called *Collatz sequences* by

$$\text{1. Choose } n_0 \in \mathbb{N} \text{ arbitrarily.} \tag{1.1}$$

$$\text{2. Define } n_j \text{ for all } j \in \mathbb{N} \text{ as follows :} \tag{1.2}$$

$$n_j := \begin{cases} \dfrac{n_{j-1}}{2}, & \text{if } n_{j-1} \text{ is even,} \\ 3n_{j-1} + 1, & \text{otherwise.} \end{cases} \tag{1.3}$$

The number n_0 is called the *starting value*. For the starting value $n_0 := 1$ we produce the following Collatz sequence: $\{n_0, n_1, n_2, \ldots\} = \{1, 4, 2, 1, 4, 2, \ldots\}$. For $n_0 = 2$ we obtain $\{2, 1, 4, 2, \ldots\}$ and for $n_0 = 7$ we have $\{7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1\}$. The *Collatz problem* consists of proving or disproving that *for all* starting values n_0 the Collatz sequence - after a finite number of steps - behaves like the sequence for $n_0 = 1$. In other words, for $n_0 \geq 3$ there is always a smallest index j_0 such that $\{n_0, n_1, \ldots, n_{j_0-2}, n_{j_0-1}, n_{j_0}, \ldots\} = \{n_0, n_1, \ldots, 4, 2, 1, \ldots\}$, $n_j > 1$ for all $j < j_0$. The *Collatz conjecture* states, that *all* Collatz sequences terminate after finitely many steps with the cycle $4, 2, 1, \ldots$ Since by definition, $3n_{j-1} + 1$ is always even in (1.3), one can immediately divide it by two and then go to the next step. If n_{j-1} is odd, say $n_{j-1} = 2k + 1$, then $n_j = 3k + 2$. The resulting sequence is called *modified Collatz sequence*. For the starting value $n_0 = 7$ we would obtain $\{7, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\}$. There are only 12 entries in comparison with 17 for $n_0 = 7$ in the standard Collatz sequence. However, there is no saving in the amount of algebraic operations. The modified Collatz sequence is a subsequence of the Collatz sequence.

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In the beginning of the paper of Collatz, [3], Collatz wrote, that in the period of 1928 to 1933 he investigated various problems related to number theory and graph theory including the $3n+1$ problem. In the same paper he wrote: “Since I could not solve the $[3n+1]$ problem, I did not publish the conjecture.” There is a warning by Guy, 1983, [4], not to treat various problems. The $3n+1$ problem is mentioned as Problem 2 without any comment.

The $3n+1$ problem - in its modified form - was transferred by Berg and Meinardus, 1994, 1995, [1, 2] into the theory of functions of a complex variable and this theory will be used by the present author to show that the Collatz conjecture is true.

2. Two linear operators U, V and their kernels. Throughout this paper we will use the following notations: $\mathbb{C}$ for the field of complex numbers, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ for the open, unit disk in $\mathbb{C}$, and $\mathcal{H}$ for the set of holomorphic functions defined on $\mathbb{D}$. Convergence in $\mathcal{H}$ is always understood pointwise in $\mathbb{D}$. If $f_n,\ f \in \mathcal{H},\ n \in \mathbb{N}$ then $f_n \to f$ means $f_n(z) \to f(z)$ for all $z \in \mathbb{D}$. For later use we introduce

$$\varphi_0(z) := 1 \text{ for all } z \in \mathbb{C}, \quad \varphi_1(z) := \frac{z}{1-z} \text{ for all } z \text{ with } z \in \mathbb{D}, \tag{2.1}$$

$$\Delta_2 := \langle \varphi_0, \varphi_1 \rangle, \tag{2.2}$$

where $\langle \cdots \rangle$ is the linear hull over $\mathbb{C}$ of the elements between the brackets. In two papers, 1994, 1995, Berg and Meinardus, [1, 2], introduced a pair of linear functional equations, which was based on the modified Collatz sequence and which has the following two essential properties.

1. The pair of functional equations can be solved by all functions $\varphi \in \Delta_2$, where Δ_2 is introduced in (2.1), (2.2).
2. The Collatz conjecture is true if and only if there are no solutions of the pair of functional equations outside of Δ_2.

Berg and Meinardus gave two equivalent formulations. One was in the form of one functional equation and the other one in the form of a system of two functional equations. Berg and Meinardus and also Wirsching, [7] showed that both forms result in the same set of solutions. We will only use this second form, and we will slightly reformulate these two functional equations in the form of a system of two linear operators starting with the definition of some auxiliar linear operators $\mathcal{H} \to \mathcal{H}$,

$$T_1[h](z) := h(z^3), \tag{2.3}$$

$$H[h](z) := \frac{1}{3z}\left(h(z^2) + \lambda h(\lambda z^2) + \lambda^2 h(\lambda^2 z^2)\right), \text{ where} \tag{2.4}$$

$$\lambda := \exp(\frac{2\pi \mathbf{i}}{3}) = \frac{1}{2}(-1 + \sqrt{3}\,\mathbf{i}). \tag{2.5}$$

The central property of λ is

$$1 + \lambda^k + \lambda^{2k} = \begin{cases} 3 & \text{if } k \text{ is a multiple of } 3, \\ 0 & \text{otherwise,} \end{cases} \qquad k = 1, 2, \ldots \tag{2.6}$$

The supposed singularity in the definition (2.4) of H at $z = 0$ is removable. To see this, an evaluation of the expression in parentheses and its derivative as well at the origin yields zero. Thus, this expression has at least a double zero at the origin. We will now define the following pair of linear operators,

$$U[h](z) := h(z) + h(-z) - 2h(z^2), \tag{2.7}$$

$$V[h](z) := 2H[h](z) - T_1[h](z) + T_1[h](-z). \tag{2.8}$$

THEOREM 2.1. *The operators U, V, just defined in (2.7), (2.8) are continuous mappings $\mathcal{H} \to \mathcal{H}$ in the following sense: Let $f_n, f \in \mathcal{H}, n \in \mathbb{N}$ and $f_n \to f$. Then,*

$$U(f_n) \to U(f), \quad V(f_n) \to V(f),$$

or, what is the same, $U(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} U(f_n)$ and the same for V.

Proof. Both operators U, V are finite linear combinations of operators of the type $F[f](z) := f(cz^k), k \in \{1,2,3\}, |c| = 1$. If $f_n \to f$ which means $f_n(z) \to f(z)$ for all $z \in \mathbb{D}$, then also $F[f_n] \to F[f]$ since $|z| < 1 \Rightarrow |cz^k| < 1$. And for the same reason, $f \in \mathcal{H} \Rightarrow F[h] \in \mathcal{H}$. □

The *kernel* of the two operators U and V is

$$\mathcal{K} := \ker(U, V) := \{h \in \mathcal{H} : U[h] = 0, V[h] = 0\}, \tag{2.9}$$

where 0 is the zero function, and the *individual kernels* of the two operators U, V are

$$\mathcal{K}_U := \ker(U) := \{h \in \mathcal{H} : U[h] = 0\}, \tag{2.10}$$

$$\mathcal{K}_V := \ker(V) := \{h \in \mathcal{H} : V[h] = 0\}. \tag{2.11}$$

If the kernels $\mathcal{K}_U$, $\mathcal{K}_V$ are already known, we could also write

$$\mathcal{K} = \{h \in \mathcal{K}_U : V[h] = 0\} = \{h \in \mathcal{K}_V : U[h] = 0\}. \tag{2.12}$$

Explicit expressions for the two individual kernels $\mathcal{K}_U$, $\mathcal{K}_V$ were already given by Berg and Meinardus in [1, 2]. The individual kernel $\mathcal{K}_V$ will be presented in Section 3. The individual kernel $\mathcal{K}_U$ will not be needed in this investigation.

The functional equations were also discussed in an overview article by Wirsching, 2000, [7]. And actually, this publication brought the functional equations to the attention of the present author. Already in 1987 Meinardus, [6], presented a functional equation in connection with the Collatz problem. In this paper, though, Meinardus calls the problem *Syracuse problem* and in the corresponding list of references other names are also used. This applies also to the article [4] by Guy.

Our main interest will be in developing tools for the determination of the kernel $\mathcal{K}$. Before we turn to the connection of the Collatz problem with the kernel $\mathcal{K}$, we will present some facts about this kernel.

LEMMA 2.2.

1. *The kernel $\mathcal{K}$, defined in (2.9), forms a vector space over $\mathbb{C}$.*
2. *$\Delta_2 \subset \mathcal{K}$, where Δ_2 is defined in (2.1), (2.2).*

Proof.

1. This follows immediately from the linearity of U and V.
2. We show that $U[\varphi_j] = 0$, $V[\varphi_j] = 0$, where φ_j are defined in (2.1), $j = 0, 1$. The remaining part follows from 1. We have $U[\varphi_0] = \varphi_0 + \varphi_0 - 2\varphi_0 = 0$, $H[h_0] = \frac{1}{3z}h_0(1+\lambda+\lambda^2) = 0$ because of (2.6) and thus, $V[\varphi_0] = -\varphi_0+\varphi_0 = 0$, $U[\varphi_1](z) = \frac{z}{1-z} - \frac{z}{1+z} - 2\frac{z^2}{1-z^2} = 0$, $H[\varphi_1](z) = \frac{z^3}{1-z^6}$, $V[\varphi_1](z) = 2\frac{z^3}{1-z^6} - \frac{z^3}{1-z^3} - \frac{z^3}{1+z^3} = 0$.

□

By the previous lemma, the kernel $\mathcal{K}$ has at least dimension two. Berg and Meinardus have already shown the following theorem.

THEOREM 2.3. *Let the domain of definition of the operators U, V be the set of all holomorphic functions defined on $\mathbb{C}$. Then, every entire member of the kernel $\mathcal{K}$ reduces to a constant.*

Proof. Berg and Meinardus, Theorem 3 in [2]. □

Since a nonconstant, holomorphic member of $\mathcal{K}$ cannot be entire - as we have seen - it must have a singularity somewhere. Whether one can show that there must be a simple pole or another type of singularity at $z = 1$ remains open here.

The relation between the kernel $\mathcal{K}$ and the Collatz conjecture is as follows.

Theorem 2.4. *The Collatz conjecture is true if and only if $\mathcal{K} = \Delta_2$. For the definitions of $\mathcal{K}$, Δ_2 see (2.9), (2.2), respectively.*

Proof. Berg and Meinardus, [1, 2], Wirsching, [7]. □

We note that φ_1 could also be replaced by $\tilde{\varphi}_1 := \frac{1}{1-z}$ because of $\tilde{\varphi}_1 = \varphi_0 + \varphi_1$.

3. The individual kernel of V. The general solution of $V[h] = 0$, denoted by h_V, is given in formula (38), p. 9 of [1] and it reads

$$\text{(3.1)}\quad \mathcal{K}_V := \left\{ h_V : h_V(z) = h_0 + \sum_{\substack{j=1\\ j\neq 3k-1}}^{\infty} h_j t_j(z), \right.$$
$$\left. \text{with } t_j(z) := z^{k_1^{(j)}} + z^{k_2^{(j)}} + \cdots + z^{k_{N_j}^{(j)}}, j =: k_1^{(j)} < k_2^{(j)} < \cdots < k_{N_j}^{(j)} \right\},$$

where N_j is the number of terms occurring in t_j. If j is even (in addition to $j \neq 3k-1$) we have $N_j = 1$ such that $t_j(z) = z^j$. The powers $z^{k_1^{(j)}} + z^{k_2^{(j)}} + \cdots + z^{k_{N_j}^{(j)}}$ which constitute t_j can be found by the following algorithm (Berg and Meinardus, p. 8-9, [1]):

Algorithm 3.1. (Forward algorithm) Algorithm for computing the exponents $k_1^{(j)}, k_2^{(j)}, \ldots, k_{N_j}^{(j)}$ starting with $k_1^{(j)} := j$.

1. Choose $j \neq 3k - 1$.
2. Put $\ell := 1$, $k_\ell := j$;
3. **while** j is odd **do**
   $j := (3j + 1)/2$; $\ell := \ell + 1$;
   $k_\ell := j$;
   end while;

For $j = 19, 22$, e. g., this algorithm yields $t_{19}(z) = z^{19} + z^{29} + z^{44}$, $t_{22} = z^{22}$. The highest power, $k_{N_j}^{(j)}$, is, therefore, always even. All other powers (if existing) are odd.

But they have more properties.

Lemma 3.2. *Let in the expansion $t_j(z) := z^{k_1^{(j)}} + z^{k_2^{(j)}} + \cdots + z^{k_{N_j}^{(j)}}$, the index j be odd and $j \neq 3k - 1$. Then, all exponents $k_2^{(j)}, k_3^{(j)}, \ldots, k_{N_j}^{(j)}$ have the property that $k_2^{(j)} + 1, k_3^{(j)} + 1, \ldots, k_{N_j}^{(j)} + 1$ are multiples of three. Which implies that these powers never occur again in the expansion h_V of the general solution of $V[h] = 0$. In addition, all powers $k_1^{(j)}, k_2^{(j)}, k_3^{(j)}, \ldots, k_{N_j-1}^{(j)}$ are odd, and $k_{N_j}^{(j)}$ is even, and $k_{N_j}^{(j)} \neq 2j$.*

Proof. Let l_1 be an odd, positive integer and $l_2 := \frac{3l_1+1}{2}$. Then l_2 and $l_2 + 1 = \frac{3l_1+1}{2} + \frac{2}{2} = 3\frac{l_1+1}{2}$ are both integers and $l_2 + 1$ is a multiple of three. In the expansion of t_j the first entry is z^j. An application of Algorithm 3.1 yields the final result. □

Theorem 3.3. *Let h_V be the power expansion of the general solution of $V[h] = 0$ as given in (3.1). Then, all powers z^k, $k \in \mathbb{N}$ appear exactly once.*

Proof. Lemma 3.2 implies that all powers can appear at most once. According to Lemma 2.2 the function $\frac{1}{1-z} = 1 + \frac{z}{1-z} = 1 + z + z^2 + \cdots$ is a special solution of $V[h] = 0$. Thus, all powers must appear exactly once. □

Since Algorithm 3.1 has no branching, it can also be executed backwards starting from any integer $m \geq 5$ such that $m + 1$ is a multiple of three, e. g. $m = 5, 8, 14, \ldots$, and stop if the newly calculated m has the property that $m+1$ is not a multiple of three. The last m determines j, since $t_j(z) = z^j + \cdots$ for all j such that $j+1$ is not a multiple of three.

ALGORITHM 3.4. (Backward algorithm) Algorithm for computing the exponents $k_{N_j}^{(j)}, k_{N_j-1}^{(j)}, \ldots, k_1^{(j)}$ in reverse order starting with $m := k_{N_j}^{(j)}$.

1. Choose $m \geq 5$ such that $m+1$ is a multiple of three.
2. Put $\ell := 1$, $k_\ell := m$;
3. **while** $m+1$ is a multiple of three **do**
   $m := (2m-1)/3$; $\ell := \ell + 1$;
   $k_\ell := m$;
   end while;
4. Adjust the indices: $k_1, k_2, \ldots, k_\ell \to j := k_\ell^{(j)}, k_{\ell-1}^{(j)}, \ldots, k_1^{(j)}$.

Starting with $m = 26$, the backward algorithm yields $17, 11, 7$. Examples with $m = 2^n$, $2 \leq n \leq 20$ can be found in Table 5.5 on p. 32.

LEMMA 3.5. *Both algorithms, the forward and the backward algorithm, introduced in 3.1, 3.4, respectively, terminate in finitely many steps.*

Proof. The backward algorithm starts with an integer $m \geq 5$ such that $m+1 = 3^k p$ where $k \geq 1$ and $p \in \mathbb{N}$ is an integer which is not a multiple of three. After k steps we arrive at $m = 2^k p - 1$ and $m+1$ is not a multiple of three. For the forward algorithm we may assume that $j+1 = 2^\kappa q$ where $q \geq 1$ is odd and not a multiple of three and $\kappa \geq 1$. The application of κ steps of the forward algorithm yields $j = 3^\kappa q - 1$ which is even. □

The proof also tells us how long the iterations of the forward and backward algorithms are. The existence of a backward algorithm shows that there is one-to-one relation between the starting value j end the terminating value $k_{N_j}^{(j)}$. This is in contrast to the Collatz sequence, where no retrieval from the last value to the starting value is possible.

The forward algorithm describes a mapping,

$$j \to j, k_2, \ldots, k_N, \quad j \neq 3k-1, \quad k \in \mathbb{N},$$

with properties listed in Lemma 3.2, whereas the backward algorithm describes a mapping

$$k_n \to k_n, k_{n-1}, \ldots, k_2, j, \quad 1 \leq n \leq N, \quad k_n = 3k+2,\ k \in \mathbb{N}.$$

Both mappings map an integer (with certain properties) to an integer vector (with certain properties), where the length of the integer vector is variable, depending on the input. Thus, it is resonable, to abbreviate the two algorithms by

out_f=alg_forward(in_f), out_b=alg_backward(in_b).

One of the essential features of the general solution h_V of $V[h] = 0$ is the fact, that in its power representation all powers z^k, $k \in \mathbb{N}$ appear exactly once. See Theorem 3.3. For this reason we redefine the series slightly by putting

$$h_j =: \eta_j + h_1,\ j \geq 3 \Rightarrow (h_j = h_1 \Leftrightarrow \eta_j = 0), \quad \eta_0 = h_0,\ \eta_1 = h_1,$$

and obtain

$$\mathcal{K}_V = \left\{ h_V : h_V(z) := \eta_0 + \eta_1 \frac{z}{1-z} + \sum_{\substack{j=3 \\ j \neq 3k-1}}^{\infty} \eta_j t_j(z),\ \text{with } t_j \text{ from above} \right\}, \tag{3.2}$$

with complex constants $\eta_j, j \ge 0, j \ne 3k-1$ such that convergence takes place. By using the backward algorithm 3.4 we can slightly reorder the series given in (3.2), such that

$$\sum_{\substack{j=3\\ j\ne 3k-1}}^{\infty} \eta_j t_j(z) = \sum_{j=3}^{\infty} \eta_{j'} z^j, \text{ where} \tag{3.3}$$

$$j' := \begin{cases} j & \text{if } j+1 \ne 3k, \\ k_\ell \text{ from Algorithm 3.4 applied to } j & \text{if } j+1 = 3k,\ k \in \mathbb{N}. \end{cases} \tag{3.4}$$

Table 3.6. Mapping $j \to j'$

j	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
j'	3	4	3	6	7	3	9	10	7	12	13	9	15	16	7	18	19	13

The mapping $j \to j'$ is not invertible. Distinct j may have the same j'. See Table 3.6.

4. The determination of the image $U(\mathcal{K}_V)$. Since $h \in \mathcal{K}_V$ has the general representation given in (3.2), we need to compute $U[t_j]$ for positive integers $j \ne 3k-1$, $k \in \mathbb{N}$, where U is defined in (2.7) and t_j in (3.1).

Lemma 4.1. *Let U and t_j, $j \ge 3$, $j \ne 3k-1$, $k \in \mathbb{N}$ be defined as in (2.7), (3.1), respectively. Then,*

$$U[t_j] = t_j(z) + t_j(-z) - 2t_j(z^2) = 2\begin{cases} t_j(z) - t_j(z^2) = z^j - z^{2j} & \textit{for even } j, \\ z^{k_{N_j}^{(j)}} - t_j(z^2) & \textit{for odd } j. \end{cases} \tag{4.1}$$

Proof. Apply the mentioned formulas. □

We will use the notation

$$\tau_j := \frac{1}{2}U[t_j], \quad j \ge 3,\ j \ne 3k-1,\ k \in \mathbb{N}, \tag{4.2}$$

where the meaning of τ_j is given on the right of (4.1), deleting the factor two. We will call the first term of τ_j the *positive term* and all other terms *negative terms*. Note, that τ_j is a polynomial in $u = z^2$. For examples see Table 5.1, p. 13. In that table, the first column (apart from the column which contains the index number j for orientation) contains the exponents which belong to the positive terms. In the remaining columns of that table, all exponents which belong to the negative terms are listed. In the following lemma we will summarize some properties of τ_j.

Lemma 4.2. *Let $j \ge 3$, $j \ne 3k-1$, $k \in \mathbb{N}$. Then τ_j as given in (4.1), (4.2), has the following properties.*

1. *Let j be even. Then $\tau_j = z^j - z^{2j}$. There are two cases: (a) $j = 6k-2$ or (b) $j = 6k$. In case (a) $2j+1$ is a multiple of three, in case (b) $2j+1$ is not a multiple of three.*
2. *Let j be odd. In this case, the exponent of the positive term, $J := k_{N_j}^{(j)}$ has the property, that J is even and that $J+1$ is a multiple of three. Therefore, it must have the form $J := k_{N_j}^{(j)} = 2(3k+1)$ for some $k \in \mathbb{N}$. The exponent of the first negative term of τ_j is $2j$, since the first term of t_j is z^j. The odd js separate into two groups: (a): $j = 3(2k-1)$ and (b): $j = 6k+1$. In case (a) we have $2j+1 = 6(2k-1)+1 = 12k-5$ and $2j+1$ is not a multiple of three. For (b) we have $2j+1 = 12k+3 = 3(4k+1)$, and $2j+1$ is a multiple of three. Let $p := 2k_s^{(j)}$, $s \ge 2$ be one of the exponents of the negative terms apart from the first one. Then $p+1$ is not a multiple of three and $p = 6k+4 = 3k'+1$ for some $k, k' \in \mathbb{N}$.*

3. *Let j be fixed. Then the exponents of all terms of τ_j are pairwise distinct.*
4. *Let $J \geq 4$ be any given, even integer. Then, there exists exactly one $j \geq 3, j \neq 3k-1, k \in \mathbb{N}$ such that the exponent of the positive term of τ_j coincides with J. Let $J \geq 6$ be any given, even integer. Then, there exists exactly one $j \geq 3, j \neq 3k-1, k \in \mathbb{N}$ such that one of the exponents of the negative terms of τ_j coincides with J.*

Proof.

1. In case (a) we have $2j = 12k-4$ or $2j+1 = 12k-3 = 3(4k-1)$ and, thus, $2j+1$ is a multiple of three. In case (b) we have $2j = 12k$ or $2j+1 = 12k+1$ and $2j+1$ is not a multiple of three.
2. Apply Lemma 3.2.
3. Clear if j is even, since $j \neq 2j$. Let j be odd. The exponents occurring in $t_j(z^2)$ are strictly increasing, thus, pairwise distinct. The exponent p of the only positive term has the property that $p+1$ is a multiple of three, whereas the second, third, etc exponent of the negative terms do not have this property (part 2 of this lemma). It remains to show that the exponents of the positive term and the first exponent of the negative term, which is $2j$ cannot agree. However, that is not possible according to the last part of Lemma 3.2.
4. Let $j \geq 3$ be a given integer. Then, there is exactly one index $j' \geq 3$ such that $j' \neq 3k-1$ and the power z^j occurs in $t_{j'}$. See formula (3.3). Thus, all $t_j(z^2)$ together, contain all even powers from six on exactly once. Since the backward algorithm 3.4 applied to $k_{N_j}^{(j)}$ retrieves j (odd, $\geq 3, j+1 \neq 3k$) uniquely, the exponents of the positive terms altogether form the even numbers from four on, and no exponent appears twice.

□

Theorem 4.3. *The expansion*

$$\frac{1}{2}U[h] = \sum_{\substack{j=3 \\ j\neq 3k-1}}^{\infty} \eta_j \tau_j, \tag{4.3}$$

where $h \in \mathcal{K}_V$, can be written in the form

$$\frac{1}{2}U[h](z) = \eta_4 z^4 + \sum_{\ell=3}^{\infty} (\eta_j - \eta_{j''})\, z^{2\ell}, \tag{4.4}$$

where $j = j(2\ell)$ denotes the row number where the exponent of the positive term is 2ℓ, and $j'' = j''(2\ell)$ denotes the row number where the exponent of the negative term is 2ℓ. For distinct values of 2ℓ the indices j will also be distinct, whereas the index j'' may appear several times if j is odd. In addition, $j \neq j''$.

Proof. This is a consequence of Lemma 4.2, part 4. The last statement follows from part 3. □

Theorem 4.4. *In order that $U[h] = 0, h \in \mathcal{K}_V$ it is necessary that in the expansion (4.4)*

$$\eta_4 = 0, \quad \eta_j - \eta_{j''} = 0 \text{ for all } 2\ell \geq 6, \tag{4.5}$$

where the mapping $2\ell \to (j, j'')$ is given in Theorem 4.3.

Proof. Follows from (4.4). □

Lemma 4.5. *Let the expansion (4.4) be given and let $\ell \neq 3k-1$. Then $j'' = \ell$.*

Proof. For $\ell \neq 3k-1$ we have $\tau_j(z) = z^{2\ell} - \cdots$, $\tau_\ell(z) = \cdots - z^{2\ell}$. Thus, $\eta_j - \eta_\ell$ is the factor of $z^{2\ell}$. □

Though the system given in (4.5) is extremely simple, it is by far not clear, that its solution is $\eta_j = 0$ for all $j \geq 3, j \neq 3k-1$, $k \in \mathbb{N}$. This would be only true, if the system does not separate into disjoint subsystems. But, because of the complicated structure of the mapping $2\ell \to (j, j'')$ this is not apriori clear.

It is good to have some examples. See Table 4.6. Note, that $\ell \neq 3k-1$ is equivalent to $2\ell = 6k$ or $2\ell = 6k+2$. Since by (4.5) we have $\eta_4 = 0$, Table 4.6 implies $\eta_3 = 0$.

TABLE 4.6. Table of j, j'' occurring in $(\eta_j - \eta_{j''})z^{2\ell}$ of formula (4.4).

2ℓ	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38
j	4	6	3	10	12	9	16	18	13	22	24	7	28	30	21	34	36	25
j''	–	3	4	3	6	7	3	9	10	7	12	13	9	15	16	7	18	19

2ℓ	40	42	44	46	48	50	52	54	56	58	60	62	64	66	68	70	72	74
j	40	42	19	46	48	33	52	54	37	58	60	27	64	66	45	70	72	49
j''	13	21	22	15	24	25	7	27	28	19	30	31	21	33	34	15	36	37

We want to study the consequences of (4.5). In particular, the question, whether all $\eta_j, \eta_{j''}$ are vanishing, is of main interest.

LEMMA 4.7. *Let (4.5) be true and let j and $2j$ have the property that $j \neq 3k-1, 2j \neq 3k-1, k \in \mathbb{N}$. Then, $\eta_j = 0$ implies that $\eta_{2^n j} = 0$ for all $n \in \mathbb{N}$.*

Proof. Let j be odd. Then

$$\eta_j\tau_j = \eta_j\big(z^{k^{(j)}_{N_j}} - (z^{2j} + \cdots)\big),$$
$$\eta_{2j}\tau_{2j} = \eta_{2j}(z^{2j} - z^{4j})$$

and the factor at z^{2j} is $\eta_j - \eta_{j''} = \eta_{2j} - \eta_j$. Let j be even. Then, $\eta_j\tau_j = \eta_j(z^j - z^{2j})$, $\eta_{2j}\tau_{2j} = \eta_{2j}(z^{2j} - z^{4j})$ and the factor at z^{2j} is again $\eta_{2j} - \eta_j$. Thus, because of (4.5), we have $\eta_j = \eta_{2j} = \cdots = \eta_{2^n j}, n \in \mathbb{N}$. □

This lemma shows already, that infinitely many of the ηs must vanish, since $\eta_3 = 0$ as already remarked. It should be noted that the two assumptions $j \neq 3k-1$ and $2j \neq 3k-1$ of Lemma 4.7 apply exactly to $j = 3k, k \in \mathbb{N}$. Thus, it is reasonable to restrict the assumptions in Lemma 4.7 to $j = 3(2k-1), k \in \mathbb{N}$.

TABLE 4.8. Some special terms $(\eta_j - \eta_{j''})z^{2\ell}$ of the expansion (4.4).

2ℓ	j	j''
326	217	163
368	163	184
184	184	61
92	61	46
46	46	15
80	15	40
40	40	13
20	13	10
10	10	3
8	3	4

In Table 4.8 we list some special terms of the expansion (4.4), which allow us, eventually, to conclude that all appearing coefficients must vanish. Basis for this table and later tables are two extensive tables 5.1, p. 13 and 5.2, p. 22, for the polynomials τ_j which are based on the forward and backward algorithms 3.1 and 3.4. And from the table data $\eta_{217} - \eta_{163} = \eta_{163} - \eta_{184} = \cdots = \eta_{10} - \eta_3 = \eta_3 - \eta_4 = 0$ we deduce $\eta_{217} = \eta_{163} = \cdots = \eta_3 = \eta_4 = 0$ since $\eta_4 = 0$. The construction of the above table has some principle meaning. It suggests an algorithm which allows us to prove that all coefficients $\eta_j, j \geq 3, j \neq 3k-1$ are vanishing.

ALGORITHM 4.9. (Annihilation algorithm) Principal form of algorithm to construct a table of type 4.8.

1. Choose any $2\ell > 8$.
   while $2\ell > 8$ **do**
2. Determine the row number j of the positive term whose exponent is 2ℓ.
3. Determine the row number j'' of the negative term whose exponent is 2ℓ.
4. Replace 2ℓ by the exponent of the positive term whose row number is j''.
   end while;

The number 2ℓ chosen in the beginning will be called *start value of the annihilation algorithm*. This algorithm creates a table with three columns, which will be named 2ℓ column, j column and j'' column. We will name this table *annihilation table*.

A look at Table 4.8 shows, that it is sufficient to compute j'' and 2ℓ with the exception of the very first j. By means of the two algorithms, forward and backward, it is easy to realize the annihilation algorithm. We present a little program in MATLAB form for creating the annihilation table.

ALGORITHM 4.10. (Annihilation algorithm) MATLAB form of algorithm to construct a table of type 4.8.

1. Choose any in_b=$2\ell > 8$; out_f=in_b; count=0;
   out_b=**alg_backward**(in_b); j(1)=out_b(length(out_b));
   while out_f(length(out_b))> 8 **do**
2. count=count+1;
3. out_b=**alg_backward**(out_f(length(out_f))/2);
4. out_f=**alg_forward**(out_b(length(out_b)));
5. twoell(count)=out_f(length(out_f)); j2dash(count)=out_f(1);
   end while;

LEMMA 4.11. *(1) The Algorithm 4.9 creates a linear system of the simple form*

$$\begin{aligned} \eta_{j_1} - \eta_{j_2} &= 0, \\ \eta_{j_2} - \eta_{j_3} &= 0, \\ \eta_{j_3} - \eta_{j_4} &= 0, \\ &\vdots \\ \eta_{j_{n-2}} - \eta_{j_{n-1}} &= 0, \\ \eta_{j_{n-1}} - \eta_{j_n} &= 0, \end{aligned}$$

where n varies with the first entry η_{j_1}. (2) If one of the entries $\eta_{j_k} = 0$, $1 \leq k \leq n$, then all other entries also vanish. (3) If one of the entries in the 2ℓ column has the form $2\ell = 2^n(2k+1)$ for $n \geq 2, k \in \mathbb{N}$, then the next entries in this column are

$$2^{n-1}(2k+1), 2^{n-2}(2k+1), \ldots, 2(2k+1).$$

(4) In the 2ℓ column of the annihilation table created by Algorithm 4.9 there are no repetitions. All occurring integers are pairwise distinct. The same applies for the j

column. In other words, the annihilation table defines a one-to-one correspondence between $\{2\ell \in \mathbb{N} : 2\ell \geq 6\}$ *and* $\{j \in \mathbb{N} : j \geq 3, j \neq 3k-1, k \in \mathbb{N}\}$.

Proof. (1) The form of the system of equations is constructed by Algorithm 4.9. (2) That one vanishing element implies the vanishing of all elements is due to the special form of the system of equations. (3) For this part see Lemma 4.7. (4) Follows from Lemma 4.2, part 4. □

If we have a look at Table 4.8, we see that the second entry in the 2ℓ column is $368 = 2^4 \cdot 23$ and consequently the next three entries are $184 = 2^3 \cdot 23, 92 = 2^2 \cdot 23$, $46 = 2 \cdot 23$. In particular, there are entries in the 2ℓ column which are smaller than the start value $2\ell = 326$.

THEOREM 4.12. *Let* $2\ell_0 > 8$ *be an arbitrary start value in Algorithm 4.9. Let the* 2ℓ *column of the annihilation table be defined by Algorithm 4.9, have the property that it always contains (at least) one value which is smaller than* $2\ell_0$. *Then, all* $\eta_j = 0,\ j \geq 3,\ j \neq 3k-1$.

Proof. The proof is by induction. If we choose $2\ell = 8$, we obtain the one row table $8, 3, 4$ and it follows $\eta_3 = 0$ since $\eta_4 = 0$. Let us now assume that all ηs which belong to all start values up to $2\ell = 2n$ are already vanishing. Let us apply the annihilation algorithm to the start value $2(n+1)$. According to our assumption the table contains an entry $< 2(n+1)$ and, thus, by Lemma 4.11 all ηs are vanishing. □

The crucial assumption here was, that, starting the algorithm with $2\ell_0$, there will always be a value of the 2ℓ column which is smaller than $2\ell_0$. However, we can even show, that the algorithm always must terminate with the last row $8, 3, 4$.

In order to show that, we can ask, whether it is possible to extend a given annihilation table beyond the top. Assume we have forgotten the first row of Table 4.8. Can we retrieve it and how. From the second row we still know that the entry of the j'' column in the first row must be 163. This refers to an exponent of a negative term with respect to $j = 163$. If we look in the corresponding annihilation table, Table 5.1, p. 13 we find for $j = 163$ three exponents of negative terms: 326, 490, 736. This opens up three possibilities for a new table which agrees with Table 4.8 from the second row on. The three possibilities for a first row are: (i) 326,217,163, (ii) 490,490,163, (iii) 736,736,163. In order to find j for the three integers, one has to look up Table 5.2, p. 22 for these numbers and find the corresponding j in the first column.

LEMMA 4.13. *Let any row of an annihilation table constructed by the algorithm 4.9 be* $[2\ell_1, j_1, j_1'']$. *Then, a continuation beyond this row is possible in several ways. It is unique if* j_1 *is even, and in this case the continuation is* $[2\ell_0, j_0, j_0'']$ *where* $j_0'' = j_1$, *and* $2\ell_0 = 4\ell_1$ *is the only exponent of the negative term of* τ_{j_1}, *and* j_0 *is the position of the positive exponent* $2\ell_0$ *and* j_0 *is odd. If* j_1 *is odd, then,* τ_{j_1} *has several (at least two) negative terms and every exponent of a negative term defines a continuation in exactly the way as described for the unique case.*

Proof. Given $[2\ell_1, j_1, j_1'']$, for the construction of the possible new rows in front of $[2\ell_1, j_1, j_1'']$ we can give the following algorithmic approach:

ALGORITHM 4.14. Algorithm for finding all predecessors of $[2\ell_1, j_1, j_1'']$.

```
if j_1 is odd
  new=2*alg_forward(j_1);
  else
  new= 4*ℓ_1;
end if;
```

The quantity `new` will be an integer vector containing all exponents of the negative term with number j_1. □

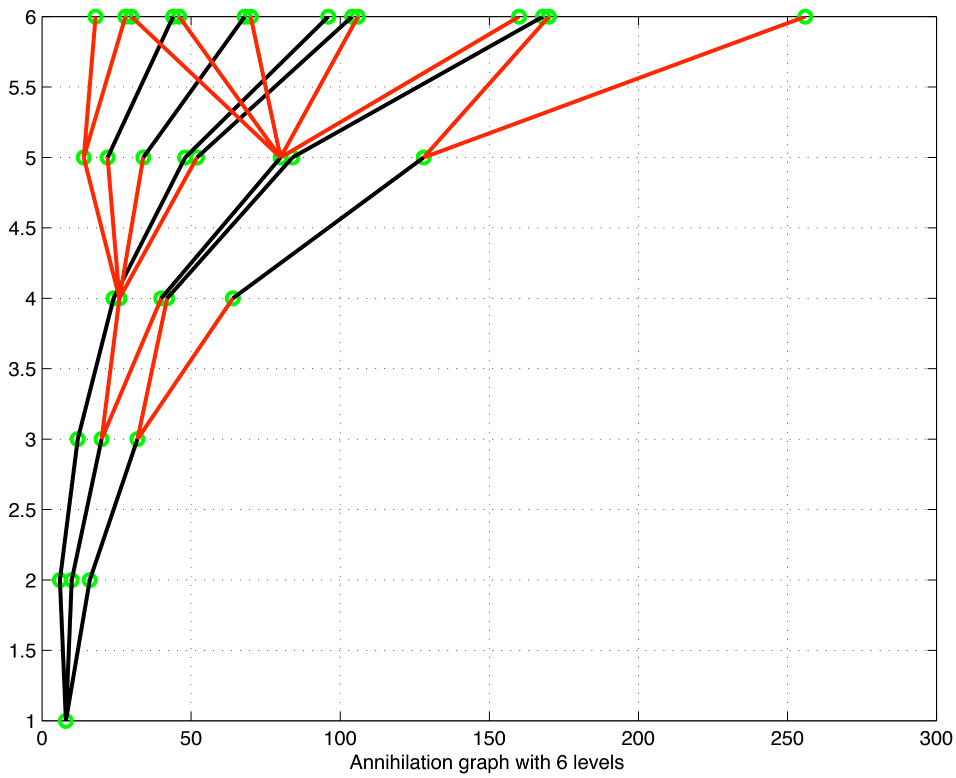

FIGURE 4.15. First six levels of annihilation graph.

Lemma 4.13 is now of principal interest, because it allows to completely describe all possible annihilation tables starting from the bottom $8, 3, 4$. If we call this the *entry of level one* the next level, number two, will be defined by the three predecessors of $8, 3, 4$ which are $6, 6, 3$; $10, 10, 3$; $16, 16, 3$. It is sufficient to keep only the first number of this triple in mind which is the exponent of the corresponding positive term. Thus, the predecessors of 8 in level one are the exponents $6, 10, 16$ in level two. We use these numbers together with the level number for the construction of a graph, which we will name *annihilation graph*. If $2n$ is an exponent in level l we connect $(2n, l)$ with $(2n_1, l+1), (2n_2, l+1), \ldots, (2n_k, l+1)$ by a straight line, reading these pairs as cartesian coordinates, and simultaneously as vertices of a graph, where $2n_1, 2n_2, \ldots, 2n_k$ are the predecessors of $2n$, defined in Algorithm 4.14. The straight connections will be the edges of the graph. And we do this for all exponents in level l. Having done this, we continue with level $l+1$. The set of all vertices $(2n, l)$ in all levels will contain all even numbers $2n \geq 6$ exactly once. For this reason, the annihilation graph is in a technical sense a tree. Going from a vertex $(2n, l)$ to a lower level is unique, whereas going upwards is in general not unique. The first six levels of this graph are shown in Figure 4.15. There is a unique correspondence between the possible annihilation tables and the vertices of the annihilation graph. An entry $2\ell, j, j''$ in row l, counted from the bottom of the annihilation table, corresponds to a vertex $(2\ell, l)$ in the graph. And if we follow the graph downwards from $(2\ell, l)$ on, we retrieve exactly the table values also downwards. For Table 4.8 we show the correspoding annihilation graph in Figure 4.16.

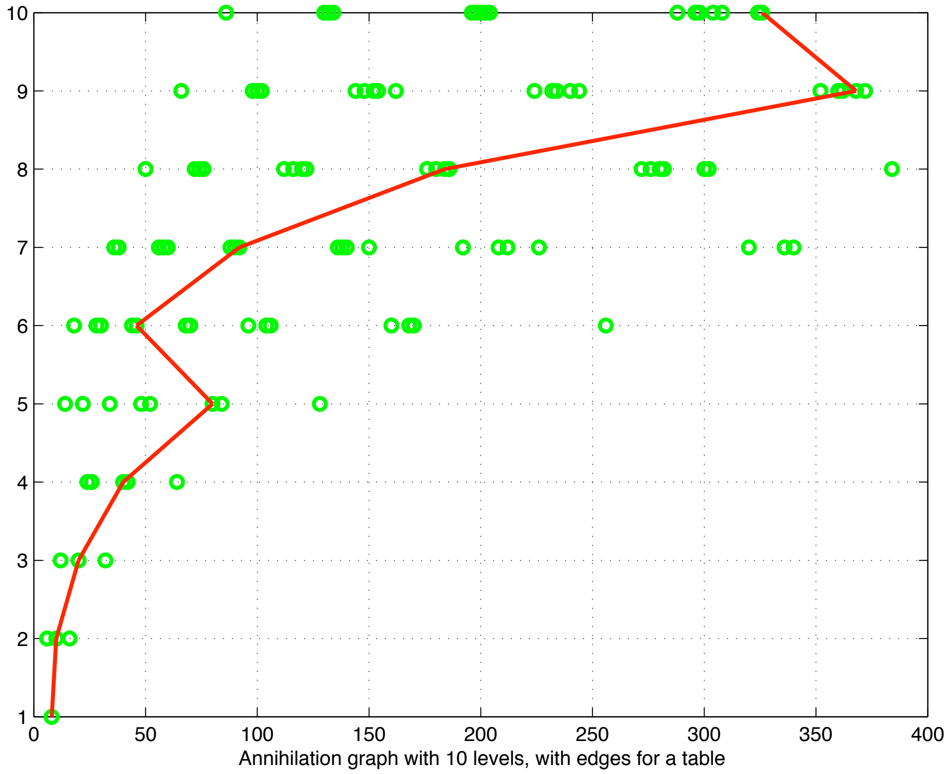

FIGURE 4.16. Annihilation graph with edges related to Table 4.8.

THEOREM 4.17. *Regardlass of the start value $2\ell > 8$, the annihilation algorithm will always end with $2\ell = 8$.*

Proof. This follows from the properties of the annihilation graph. Every start value $2\ell > 8$ defines a vertex of the graph and because of the tree structure it always ends at $2\ell = 8$. □

CONCLUSION 4.18. *Theorem 4.12 is valid without any restriction. Thus, $U[h] = 0$ for $h \in \mathcal{K}_V$ implies*

$$\mathcal{K} = \Delta_2,$$

where these quantities are defined in (2.9), (2.2).

Proof. Theorem 4.12 requires a 2ℓ value somewhere in the 2ℓ column of the annihilation table which is less than the start value $2\ell_0$. But this follows from Theorem 4.17. □

If we have a look at Theorem 2.4, and compare it with Conclusion 4.18, we see that the Collatz conjecture is true. This result is essentially based on the analytic approach of the Collatz $3n + 1$ problem by Berg and Meinardus, [1, 2].

5. A collection of longer tables.

TABLE 5.1. Table of exponents of polynomials τ_j ordered with respect to j. The numbers in the second column represent the exponents belonging to the positive term of τ_j, the numbers in the third and later columns represent the exponents of the negative terms of τ_j. Entries x with $x + 1 = 3k$ appear in red.

$j = 3$	8	6	10	16				
4	4	8						
6	6	12						
7	26	14	22	34	52			
9	14	18	28					
10	10	20						
12	12	24						
13	20	26	40					
15	80	30	46	70	106	160		
16	16	32						
18	18	36						
19	44	38	58	88				
21	32	42	64					
22	22	44						
24	24	48						
25	38	50	76					
27	62	54	82	124				
28	28	56						
30	30	60						
31	242	62	94	142	214	322	484	
33	50	66	100					
34	34	68						
36	36	72						
37	56	74	112					
39	134	78	118	178	268			
40	40	80						
42	42	84						
43	98	86	130	196				
45	68	90	136					
46	46	92						
48	48	96						
49	74	98	148					
51	116	102	154	232				
52	52	104						
54	54	108						
55	188	110	166	250	376			
57	86	114	172					
58	58	116						
60	60	120						
61	92	122	184					
63	728	126	190	286	430	646	970	1456
64	64	128						
66	66	132						
67	152	134	202	304				
69	104	138	208					
70	70	140						
72	72	144						

73	110	146	220						
75	170	150	226	340					
76	76	152							
78	78	156							
79	404	158	238	358	538	808			
81	122	162	244						
82	82	164							
84	84	168							
85	128	170	256						
87	296	174	262	394	592				
88	88	176							
90	90	180							
91	206	182	274	412					
93	140	186	280						
94	94	188							
96	96	192							
97	146	194	292						
99	224	198	298	448					
100	100	200							
102	102	204							
103	350	206	310	466	700				
105	158	210	316						
106	106	212							
108	108	216							
109	164	218	328						
111	566	222	334	502	754	1132			
112	112	224							
114	114	228							
115	260	230	346	520					
117	176	234	352						
118	118	236							
120	120	240							
121	182	242	364						
123	278	246	370	556					
124	124	248							
126	126	252							
127	2186	254	382	574	862	1294	1942	2914	4372
129	194	258	388						
130	130	260							
132	132	264							
133	200	266	400						
135	458	270	406	610	916				
136	136	272							
138	138	276							
139	314	278	418	628					
141	212	282	424						
142	142	284							
144	144	288							
145	218	290	436						
147	332	294	442	664					
148	148	296							
150	150	300							
151	512	302	454	682	1024				
153	230	306	460						

154	154	308					
156	156	312					
157	236	314	472				
159	1214	318	478	718	1078	1618	2428
160	160	320					
162	162	324					
163	368	326	490	736			
165	248	330	496				
166	166	332					
168	168	336					
169	254	338	508				
171	386	342	514	772			
172	172	344					
174	174	348					
175	890	350	526	790	1186	1780	
177	266	354	532				
178	178	356					
180	180	360					
181	272	362	544				
183	620	366	550	826	1240		
184	184	368					
186	186	372					
187	422	374	562	844			
189	284	378	568				
190	190	380					
192	192	384					
193	290	386	580				
195	440	390	586	880			
196	196	392					
198	198	396					
199	674	398	598	898	1348		
201	302	402	604				
202	202	404					
204	204	408					
205	308	410	616				
207	1052	414	622	934	1402	2104	
208	208	416					
210	210	420					
211	476	422	634	952			
213	320	426	640				
214	214	428					
216	216	432					
217	326	434	652				
219	494	438	658	988			
220	220	440					
222	222	444					
223	1700	446	670	1006	1510	2266	3400
225	338	450	676				
226	226	452					
228	228	456					
229	344	458	688				
231	782	462	694	1042	1564		
232	232	464					
234	234	468					

235	530	470	706	1060						
237	356	474	712							
238	238	476								
240	240	480								
241	362	482	724							
243	548	486	730	1096						
244	244	488								
246	246	492								
247	836	494	742	1114	1672					
249	374	498	748							
250	250	500								
252	252	504								
253	380	506	760							
255	6560	510	766	1150	1726	2590	3886	5830	8746	13120
256	256	512								
258	258	516								
259	584	518	778	1168						
261	392	522	784							
262	262	524								
264	264	528								
265	398	530	796							
267	602	534	802	1204						
268	268	536								
270	270	540								
271	1376	542	814	1222	1834	2752				
273	410	546	820							
274	274	548								
276	276	552								
277	416	554	832							
279	944	558	838	1258	1888					
280	280	560								
282	282	564								
283	638	566	850	1276						
285	428	570	856							
286	286	572								
288	288	576								
289	434	578	868							
291	656	582	874	1312						
292	292	584								
294	294	588								
295	998	590	886	1330	1996					
297	446	594	892							
298	298	596								
300	300	600								
301	452	602	904							
303	1538	606	910	1366	2050	3076				
304	304	608								
306	306	612								
307	692	614	922	1384						
309	464	618	928							
310	310	620								
312	312	624								
313	470	626	940							
315	710	630	946	1420						

316	316	632						
318	318	636						
319	3644	638	958	1438	2158	3238	4858	7288
321	482	642	964					
322	322	644						
324	324	648						
325	488	650	976					
327	1106	654	982	1474	2212			
328	328	656						
330	330	660						
331	746	662	994	1492				
333	500	666	1000					
334	334	668						
336	336	672						
337	506	674	1012					
339	764	678	1018	1528				
340	340	680						
342	342	684						
343	1160	686	1030	1546	2320			
345	518	690	1036					
346	346	692						
348	348	696						
349	524	698	1048					
351	2672	702	1054	1582	2374	3562	5344	
352	352	704						
354	354	708						
355	800	710	1066	1600				
357	536	714	1072					
358	358	716						
360	360	720						
361	542	722	1084					
363	818	726	1090	1636				
364	364	728						
366	366	732						
367	1862	734	1102	1654	2482	3724		
369	554	738	1108					
370	370	740						
372	372	744						
373	560	746	1120					
375	1268	750	1126	1690	2536			
376	376	752						
378	378	756						
379	854	758	1138	1708				
381	572	762	1144					
382	382	764						
384	384	768						
385	578	770	1156					
387	872	774	1162	1744				
388	388	776						
390	390	780						
391	1322	782	1174	1762	2644			
393	590	786	1180					
394	394	788						
396	396	792						

397	596	794	1192					
399	2024	798	1198	1798	2698	4048		
400	400	800						
402	402	804						
403	908	806	1210	1816				
405	608	810	1216					
406	406	812						
408	408	816						
409	614	818	1228					
411	926	822	1234	1852				
412	412	824						
414	414	828						
415	3158	830	1246	1870	2806	4210	6316	
417	626	834	1252					
418	418	836						
420	420	840						
421	632	842	1264					
423	1430	846	1270	1906	2860			
424	424	848						
426	426	852						
427	962	854	1282	1924				
429	644	858	1288					
430	430	860						
432	432	864						
433	650	866	1300					
435	980	870	1306	1960				
436	436	872						
438	438	876						
439	1484	878	1318	1978	2968			
441	662	882	1324					
442	442	884						
444	444	888						
445	668	890	1336					
447	5102	894	1342	2014	3022	4534	6802	10204
448	448	896						
450	450	900						
451	1016	902	1354	2032				
453	680	906	1360					
454	454	908						
456	456	912						
457	686	914	1372					
459	1034	918	1378	2068				
460	460	920						
462	462	924						
463	2348	926	1390	2086	3130	4696		
465	698	930	1396					
466	466	932						
468	468	936						
469	704	938	1408					
471	1592	942	1414	2122	3184			
472	472	944						
474	474	948						
475	1070	950	1426	2140				
477	716	954	1432					

478	478	956									
480	480	960									
481	722	962	1444								
483	1088	966	1450	2176							
484	484	968									
486	486	972									
487	1646	974	1462	2194	3292						
489	734	978	1468								
490	490	980									
492	492	984									
493	740	986	1480								
495	2510	990	1486	2230	3346	5020					
496	496	992									
498	498	996									
499	1124	998	1498	2248							
501	752	1002	1504								
502	502	1004									
504	504	1008									
505	758	1010	1516								
507	1142	1014	1522	2284							
508	508	1016									
510	510	1020									
511	19682	1022	1534	2302	3454	5182	7774	11662	17494	26242	39364
513	770	1026	1540								
514	514	1028									
516	516	1032									
517	776	1034	1552								
519	1754	1038	1558	2338	3508						
520	520	1040									
522	522	1044									
523	1178	1046	1570	2356							
525	788	1050	1576								
526	526	1052									
528	528	1056									
529	794	1058	1588								
531	1196	1062	1594	2392							
532	532	1064									
534	534	1068									
535	1808	1070	1606	2410	3616						
537	806	1074	1612								
538	538	1076									
540	540	1080									
541	812	1082	1624								
543	4130	1086	1630	2446	3670	5506	8260				
544	544	1088									
546	546	1092									
547	1232	1094	1642	2464							
549	824	1098	1648								
550	550	1100									
552	552	1104									
553	830	1106	1660								
555	1250	1110	1666	2500							
556	556	1112									
558	558	1116									

559	2834	1118	1678	2518	3778	5668			
561	842	1122	1684						
562	562	1124							
564	564	1128							
565	848	1130	1696						
567	1916	1134	1702	2554	3832				
568	568	1136							
570	570	1140							
571	1286	1142	1714	2572					
573	860	1146	1720						
574	574	1148							
576	576	1152							
577	866	1154	1732						
579	1304	1158	1738	2608					
580	580	1160							
582	582	1164							
583	1970	1166	1750	2626	3940				
585	878	1170	1756						
586	586	1172							
588	588	1176							
589	884	1178	1768						
591	2996	1182	1774	2662	3994	5992			
592	592	1184							
594	594	1188							
595	1340	1190	1786	2680					
597	896	1194	1792						
598	598	1196							
600	600	1200							
601	902	1202	1804						
603	1358	1206	1810	2716					
604	604	1208							
606	606	1212							
607	4616	1214	1822	2734	4102	6154	9232		
609	914	1218	1828						
610	610	1220							
612	612	1224							
613	920	1226	1840						
615	2078	1230	1846	2770	4156				
616	616	1232							
618	618	1236							
619	1394	1238	1858	2788					
621	932	1242	1864						
622	622	1244							
624	624	1248							
625	938	1250	1876						
627	1412	1254	1882	2824					
628	628	1256							
630	630	1260							
631	2132	1262	1894	2842	4264				
633	950	1266	1900						
634	634	1268							
636	636	1272							
637	956	1274	1912						
639	10934	1278	1918	2878	4318	6478	9718	14578	21868

640	640	1280							
642	642	1284							
643	1448	1286	1930	2896					
645	968	1290	1936						
646	646	1292							
648	648	1296							
649	974	1298	1948						
651	1466	1302	1954	2932					
652	652	1304							
654	654	1308							
655	3320	1310	1966	2950	4426	6640			
657	986	1314	1972						
658	658	1316							
660	660	1320							
661	992	1322	1984						
663	2240	1326	1990	2986	4480				
664	664	1328							
666	666	1332							
667	1502	1334	2002	3004					
669	1004	1338	2008						
670	670	1340							
672	672	1344							
673	1010	1346	2020						
675	1520	1350	2026	3040					
676	676	1352							
678	678	1356							
679	2294	1358	2038	3058	4588				
681	1022	1362	2044						
682	682	1364							
684	684	1368							
685	1028	1370	2056						
687	3482	1374	2062	3094	4642	6964			
688	688	1376							
690	690	1380							
691	1556	1382	2074	3112					
693	1040	1386	2080						
694	694	1388							
696	696	1392							
697	1046	1394	2092						
699	1574	1398	2098	3148					
700	700	1400							
702	702	1404							
703	8018	1406	2110	3166	4750	7126	10690	16036	
705	1058	1410	2116						
706	706	1412							
708	708	1416							
709	1064	1418	2128						
711	2402	1422	2134	3202	4804				
712	712	1424							
714	714	1428							
715	1610	1430	2146	3220					
717	1076	1434	2152						
718	718	1436							
720	720	1440							

721	1082	1442	2164				
723	1628	1446	2170	3256			
724	724	1448					
726	726	1452					
727	2456	1454	2182	3274	4912		
729	1094	1458	2188				
730	730	1460					
732	732	1464					
733	1100	1466	2200				
735	5588	1470	2206	3310	4966	7450	11176
736	736	1472					
738	738	1476					
739	1664	1478	2218	3328			
741	1112	1482	2224				
742	742	1484					
744	744	1488					
745	1118	1490	2236				
747	1682	1494	2242	3364			
748	748	1496					
750	750	1500					
751	3806	1502	2254	3382	5074	7612	

TABLE 5.2. Table of exponents of polynomials τ_j ordered with respect to the exponents of the only positive term of τ_j. The first column contains the number j which does not belong to the list of exponents. The numbers in the second column represent the exponents belonging to the positive term of τ_j, the numbers in the third and later columns represent the exponents of the negative terms. Entries x with $x+1=3k$ appear in red.

$j=4$	4	8			
6	6	12			
3	8	6	10	16	
10	10	20			
12	12	24			
9	14	18	28		
16	16	32			
18	18	36			
13	20	26	40		
22	22	44			
24	24	48			
7	26	14	22	34	52
28	28	56			
30	30	60			
21	32	42	64		
34	34	68			
36	36	72			
25	38	50	76		
40	40	80			
42	42	84			
19	44	38	58	88	
46	46	92			
48	48	96			
33	50	66	100		
52	52	104			

54	54	108				
37	56	74	112			
58	58	116				
60	60	120				
27	62	54	82	124		
64	64	128				
66	66	132				
45	68	90	136			
70	70	140				
72	72	144				
49	74	98	148			
76	76	152				
78	78	156				
15	80	30	46	70	106	160
82	82	164				
84	84	168				
57	86	114	172			
88	88	176				
90	90	180				
61	92	122	184			
94	94	188				
96	96	192				
43	98	86	130	196		
100	100	200				
102	102	204				
69	104	138	208			
106	106	212				
108	108	216				
73	110	146	220			
112	112	224				
114	114	228				
51	116	102	154	232		
118	118	236				
120	120	240				
81	122	162	244			
124	124	248				
126	126	252				
85	128	170	256			
130	130	260				
132	132	264				
39	134	78	118	178	268	
136	136	272				
138	138	276				
93	140	186	280			
142	142	284				
144	144	288				
97	146	194	292			
148	148	296				
150	150	300				
67	152	134	202	304		
154	154	308				
156	156	312				
105	158	210	316			
160	160	320				

162	162	324					
109	164	218	328				
166	166	332					
168	168	336					
75	170	150	226	340			
172	172	344					
174	174	348					
117	176	234	352				
178	178	356					
180	180	360					
121	182	242	364				
184	184	368					
186	186	372					
55	188	110	166	250	376		
190	190	380					
192	192	384					
129	194	258	388				
196	196	392					
198	198	396					
133	200	266	400				
202	202	404					
204	204	408					
91	206	182	274	412			
208	208	416					
210	210	420					
141	212	282	424				
214	214	428					
216	216	432					
145	218	290	436				
220	220	440					
222	222	444					
99	224	198	298	448			
226	226	452					
228	228	456					
153	230	306	460				
232	232	464					
234	234	468					
157	236	314	472				
238	238	476					
240	240	480					
31	242	62	94	142	214	322	484
244	244	488					
246	246	492					
165	248	330	496				
250	250	500					
252	252	504					
169	254	338	508				
256	256	512					
258	258	516					
115	260	230	346	520			
262	262	524					
264	264	528					
177	266	354	532				
268	268	536					

270	270	540			
181	272	362	544		
274	274	548			
276	276	552			
123	278	246	370	556	
280	280	560			
282	282	564			
189	284	378	568		
286	286	572			
288	288	576			
193	290	386	580		
292	292	584			
294	294	588			
87	296	174	262	394	592
298	298	596			
300	300	600			
201	302	402	604		
304	304	608			
306	306	612			
205	308	410	616		
310	310	620			
312	312	624			
139	314	278	418	628	
316	316	632			
318	318	636			
213	320	426	640		
322	322	644			
324	324	648			
217	326	434	652		
328	328	656			
330	330	660			
147	332	294	442	664	
334	334	668			
336	336	672			
225	338	450	676		
340	340	680			
342	342	684			
229	344	458	688		
346	346	692			
348	348	696			
103	350	206	310	466	700
352	352	704			
354	354	708			
237	356	474	712		
358	358	716			
360	360	720			
241	362	482	724		
364	364	728			
366	366	732			
163	368	326	490	736	
370	370	740			
372	372	744			
249	374	498	748		
376	376	752			

378	378	756				
253	380	506	760			
382	382	764				
384	384	768				
171	386	342	514	772		
388	388	776				
390	390	780				
261	392	522	784			
394	394	788				
396	396	792				
265	398	530	796			
400	400	800				
402	402	804				
79	404	158	238	358	538	808
406	406	812				
408	408	816				
273	410	546	820			
412	412	824				
414	414	828				
277	416	554	832			
418	418	836				
420	420	840				
187	422	374	562	844		
424	424	848				
426	426	852				
285	428	570	856			
430	430	860				
432	432	864				
289	434	578	868			
436	436	872				
438	438	876				
195	440	390	586	880		
442	442	884				
444	444	888				
297	446	594	892			
448	448	896				
450	450	900				
301	452	602	904			
454	454	908				
456	456	912				
135	458	270	406	610	916	
460	460	920				
462	462	924				
309	464	618	928			
466	466	932				
468	468	936				
313	470	626	940			
472	472	944				
474	474	948				
211	476	422	634	952		
478	478	956				
480	480	960				
321	482	642	964			
484	484	968				

486	486	972				
325	488	650	976			
490	490	980				
492	492	984				
219	494	438	658	988		
496	496	992				
498	498	996				
333	500	666	1000			
502	502	1004				
504	504	1008				
337	506	674	1012			
508	508	1016				
510	510	1020				
151	512	302	454	682	1024	
514	514	1028				
516	516	1032				
345	518	690	1036			
520	520	1040				
522	522	1044				
349	524	698	1048			
526	526	1052				
528	528	1056				
235	530	470	706	1060		
532	532	1064				
534	534	1068				
357	536	714	1072			
538	538	1076				
540	540	1080				
361	542	722	1084			
544	544	1088				
546	546	1092				
243	548	486	730	1096		
550	550	1100				
552	552	1104				
369	554	738	1108			
556	556	1112				
558	558	1116				
373	560	746	1120			
562	562	1124				
564	564	1128				
111	566	222	334	502	754	1132
568	568	1136				
570	570	1140				
381	572	762	1144			
574	574	1148				
576	576	1152				
385	578	770	1156			
580	580	1160				
582	582	1164				
259	584	518	778	1168		
586	586	1172				
588	588	1176				
393	590	786	1180			
592	592	1184				

594	594	1188			
397	596	794	1192		
598	598	1196			
600	600	1200			
267	602	534	802	1204	
604	604	1208			
606	606	1212			
405	608	810	1216		
610	610	1220			
612	612	1224			
409	614	818	1228		
616	616	1232			
618	618	1236			
183	620	366	550	826	1240
622	622	1244			
624	624	1248			
417	626	834	1252		
628	628	1256			
630	630	1260			
421	632	842	1264		
634	634	1268			
636	636	1272			
283	638	566	850	1276	
640	640	1280			
642	642	1284			
429	644	858	1288		
646	646	1292			
648	648	1296			
433	650	866	1300		
652	652	1304			
654	654	1308			
291	656	582	874	1312	
658	658	1316			
660	660	1320			
441	662	882	1324		
664	664	1328			
666	666	1332			
445	668	890	1336		
670	670	1340			
672	672	1344			
199	674	398	598	898	1348
676	676	1352			
678	678	1356			
453	680	906	1360		
682	682	1364			
684	684	1368			
457	686	914	1372		
688	688	1376			
690	690	1380			
307	692	614	922	1384	
694	694	1388			
696	696	1392			
465	698	930	1396		
700	700	1400			

702	702	1404							
469	704	938	1408						
706	706	1412							
708	708	1416							
315	710	630	946	1420					
712	712	1424							
714	714	1428							
477	716	954	1432						
718	718	1436							
720	720	1440							
481	722	962	1444						
724	724	1448							
726	726	1452							
63	728	126	190	286	430	646	970	1456	
730	730	1460							
732	732	1464							
489	734	978	1468						
736	736	1472							
738	738	1476							
493	740	986	1480						
742	742	1484							
744	744	1488							
331	746	662	994	1492					
748	748	1496							
750	750	1500							
501	752	1002	1504						
754	754	1508							
756	756	1512							
505	758	1010	1516						
760	760	1520							
762	762	1524							
339	764	678	1018	1528					
766	766	1532							
768	768	1536							
513	770	1026	1540						
772	772	1544							
774	774	1548							
517	776	1034	1552						
778	778	1556							
780	780	1560							
231	782	462	694	1042	1564				
784	784	1568							
786	786	1572							
525	788	1050	1576						
790	790	1580							
792	792	1584							
529	794	1058	1588						
796	796	1592							
798	798	1596							
355	800	710	1066	1600					
802	802	1604							
804	804	1608							
537	806	1074	1612						
808	808	1616							

810	810	1620				
541	812	1082	1624			
814	814	1628				
816	816	1632				
363	818	726	1090	1636		
820	820	1640				
822	822	1644				
549	824	1098	1648			
826	826	1652				
828	828	1656				
553	830	1106	1660			
832	832	1664				
834	834	1668				
247	836	494	742	1114	1672	
838	838	1676				
840	840	1680				
561	842	1122	1684			
844	844	1688				
846	846	1692				
565	848	1130	1696			
850	850	1700				
852	852	1704				
379	854	758	1138	1708		
856	856	1712				
858	858	1716				
573	860	1146	1720			
862	862	1724				
864	864	1728				
577	866	1154	1732			
868	868	1736				
870	870	1740				
387	872	774	1162	1744		
874	874	1748				
876	876	1752				
585	878	1170	1756			
880	880	1760				
882	882	1764				
589	884	1178	1768			
886	886	1772				
888	888	1776				
175	890	350	526	790	1186	1780
892	892	1784				
894	894	1788				
597	896	1194	1792			
898	898	1796				
900	900	1800				
601	902	1202	1804			
904	904	1808				
906	906	1812				
403	908	806	1210	1816		
910	910	1820				
912	912	1824				
609	914	1218	1828			
916	916	1832				

918	918	1836			
613	920	1226	1840		
922	922	1844			
924	924	1848			
411	926	822	1234	1852	
928	928	1856			
930	930	1860			
621	932	1242	1864		
934	934	1868			
936	936	1872			
625	938	1250	1876		
940	940	1880			
942	942	1884			
279	944	558	838	1258	1888
946	946	1892			
948	948	1896			
633	950	1266	1900		
952	952	1904			
954	954	1908			
637	956	1274	1912		
958	958	1916			
960	960	1920			
427	962	854	1282	1924	
964	964	1928			
966	966	1932			
645	968	1290	1936		
970	970	1940			
972	972	1944			
649	974	1298	1948		
976	976	1952			
978	978	1956			
435	980	870	1306	1960	
982	982	1964			
984	984	1968			
657	986	1314	1972		
988	988	1976			
990	990	1980			
661	992	1322	1984		
994	994	1988			
996	996	1992			
295	998	590	886	1330	1996
1000	1000	2000			
1002	1002	2004			

TABLE 5.5. Exponents of the general power expansion of h satisfying $V[h] = 0$ ordered backwards, starting with powers of two. Entries m with $m + 1 = 3k$ in red.

4			
8	5	3	
16			
32	21		
64			
128	85		
256			
512	341	227	151
1024			
2048	1365		
4096			
8192	5461		
16384			
32768	21845	14563	
65536			
131072	87381		
262144			
524288	349525		
1048576			

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[1] The German title as used here was handwritten by Collatz on a copy of the Chinese paper.

[2] At the time of the submission (May 25, 2011), this book, announced by the publisher, was not yet available to the author.